getting rectilinear motion are more convenient in practice. Many special linkworks have been constructed to describe different curves, such as conics, cardioids, limaçons, lemniscates, cissoids, etc. The simplest conicograph contains seven bars. Linkages have been applied to complex variables and elliptic functions, and to realizing numerous algebraic transformations.[1]

The study of the curves described by a linkwork can most readily be made by the use of polar coördinates, which may later be transformed into Cartesian.[2] In general, a three-bar linkwork describes a sextic curve, and in general the curves described by any linkwork turn out to be algebraic. This and its converse were stated by Peaucellier[3] from general reasoning, which did not satisfy Sylvester, who outlined a proof. It remained for Kempe to prove explicitly: "A linkwork can be found to describe any given algebraic curve." There is no space here to outline his simple and elegant proof.[4] The method is much more complicated than most of the special linkworks devised for important curves, but it is of theoretical value. The opposite theorem is also true, that no transcendental curve can be described by a linkwork. These results definitely show us the large possibilities of linkworks, and also their limitations.

THE PROBABLE RANK IN A LARGE CLASS OF A STUDENT OF GIVEN RANK IN A SMALL CLASS.

By L. D. AMES, University of Missouri.

In a certain statistical investigation it was desirable to compare students in different classes on the basis of their ranks in their respective classes. To reduce the two rankings to a common unit we seek to know the probable rank in a very large class of a student who ranks kth in a class of n students.

For example, we may wish to know how a given class of students who have had a certain definite type of training compare in their subsequent work with the average of students with whom they may compete. We follow these particular students into various other classes and find, for example, that one of them ranks second in a class of nine, another ranks sixth in a class of ten, etc. We wish to assign a numerical value to these rankings. Does the second student in a class of four probably rank, other things being equal, on a par with the twenty-second student in a class of forty-four?

[1] See Emch, *An Introduction to Projective Geometry and Its Applications*, Wiley, 1905, for a chapter on linkage transformations and references to original papers.

[2] For examples of the methods of attack, see F. Dingeldey, *Über die Erzeugung von Kurven vierter Ordnung durch Bewegungsmechanismen*, Teubner, 1885, Chapter III. Also Königs, *Leçons de cinématique*, Paris, 1897, Chap. XI.

[3] *Note sur une question de géometrie de compas, Nouvelles Annales*, 2e série, XII, p. 71, 1873. This, by the way, is Peaucellier's first published description of his invention.

[4] *On a General Method of Describing Curves of the nth Degree by a Linkwork. Proc. of London Math. Soc.*, 1876, VI, pp. 213–216. Also given in Königs, *op. cit.*, pp. 269–273.

To answer this question we suppose that a very large number of students are equally distributed along a line of given length. A class of n students is picked at random from this large number. The kth student of this class, counting from one end, is noted, and his distance from that end. This is repeated a large number of times and the distances averaged. What is the probable value of this average?

Clearly the second student in a class of three, and the fifth student in a class of nine stand at the middle of their respective classes, or at 1/2 on a unit scale. The result reached below agrees with this conclusion. In a class of three the first student would be ranked as 1/4, the second as 2/4, the third as 3/4. In general, if a scale of any convenient length is divided into equal parts one greater in number than the number of students in the class, and the students are ranked on the scale at the points of division in the order of their rank in the class, the end points of the scale not being used, then their distances from one end will represent their probable ranks in a very large class.

To avoid irrelevant assumptions we state the problem in abstract form. If n points be independently chosen at random on the interval from $x = 0$ to $x = 1$, find the probable distance from $x = 0$ to the kth point counting from that end. By the words "chosen at random" we mean that the probability that any specified point will be taken from any interval is the same as the probability that it will be taken from any equal interval.

Let us think of the points as chosen successively. The probability that the first point chosen will be chosen from the interval from x to $x + \Delta x$ is precisely Δx. The probability that the next $k - 1$ points chosen will be chosen from the interval from 0 to x is x^{k-1}. The probability that the last $n - k$ points chosen will be chosen from the interval from x to 1 is $(1 - x)^{n-k}$. The probability that all of these things will happen together is the product of their separate probabilities. But, as any one of the n points is equally likely to be the kth point from the origin, we must multiply this by n, and as for each one of these cases the number of different possibilities of selecting the points which fall in the interval from 0 to x is the number of combinations of $n - 1$ things taken $k - 1$ at a time, the probability that the kth point from the origin will lie in the interval from x to $x + \Delta x$ is the product of these probabilities, or

$$n \cdot {}_{n-1}C_{k-1} x^{k-1}(1 - x)^{n-k}\Delta x.$$

If now we multiply this probability by x, the distance from the origin to a point of the interval, add these terms for all the intervals, take the limit as Δx approaches zero and divide by the total probability which is 1, we have as the probable average distance from the origin to the kth point counting from the origin the definite integral

$$n \cdot {}_{n-1}C_{k-1} \int_{x=0}^{x=1} x^k(1 - x)^{n-k}dx.$$

Successive integration by parts gives the probable distance as $k/(n + 1)$.

If both classes are supposed finite, we may ask what is the probable rank, K, in a class of N students of the kth student in a class of n students. We have $K/(N+1) = k/(n+1)$, or $K = k(N+1)/(n+1)$. Thus, for example, the second student in a class of four, according to the above assumptions would stand at a distance of 2/5 from the origin, and would rank the same as the eighteenth student in a class of 44.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

The Development of the Arabic Numerals in Europe Exhibited in Sixty-four Tables. By G. F. HILL. Oxford, Clarendon Press, 1915.

In *Archæologia*, Vol. LXII, Mr. G. F. Hill, keeper of the department of medals and coins in the British Museum, published a series of fifty-one tables of Hindu-Arabic numerals as they appeared in Mss. and on monuments, coins, seals, medals, brasses, and paintings, as well as a few forms from printed works. This article has been of great value to all who are interested in the exact dating of mediæval manuscripts, for the work furnishes, where numerals are used, definite checks upon the time when a manuscript was written. Further and more particularly, the work is of interest to students of the history of science for it shows in graphical form the slow but sure progress of the Hindu arithmetic through Europe. Hearty welcome, then, is given to the present publication of the Oxford Press, in which appear not only the tables of the earlier work, but also additional tables of forms more recently located.

In general the forms included are only those which antedate 1500 A.D., for by that time the new arithmetic had become well-nigh universal. Parenthetically it is of interest to note, however, that as late as 1540, Köbel, in Germany, published an arithmetic wholly, except paging, in Roman numerals.

The earliest European forms are doubtless those found in the *Codex Vigilanus*, written 976 A.D. in the monastery of Albelda near Logrono in Spain. A second Spanish manuscript of about the same date, not described by Mr. Hill, also contains similar forms, and facsimiles. Both are to appear in the next issue of Professor John M. Burnam's *Palæographia iberica*. The numerals appear to have been known in Syria in 662 A.D., for a Syrian Bishop, Severus Sebokt, of a monastery on the Euphrates, refers in a work of that time to the "easy method of their, *i. e.*, the Hindus', calculations and of their computations which surpasses words. I mean that made with nine symbols," referring certainly to the Hindu system of numerals with the zero.

Interesting also is the fact that Mr. Hill finds that the forms which afford the best criteria for dating are those for 2, 4, and 7, while 5, "the most freakish of all figures," comes next. In probably the earliest translation of an Arabic treatise on the Hindu arithmetic, the *Algoritmi de numero Indorum* (published by Bon-